The Leibniz integral rule

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1 Mean Value Theorem

Claim:

Let $f(x) \in C^{[a,b]}(x)$, where $C^{[a,b]}(x)$ is the set of all non-singular functions of a real variable x that are continuous on an interval $[a, b]$ of $\mathbb{R}$. Then for some $x_0 \in [a, b]$, $f(x_0) = f_{\text{avg}}[a, b]$.

Proof:

First, note that

$$f_{\text{avg}}[a, b] = \frac{\int_a^b f(x)dx}{(b-a)}. \tag{1}$$

Since in one-variable we can interpret a definite integral as the (signed) area bounded between the curve defined by $f(x)$ and the x-axis (Riemann sum), we can put a bound on the integral. Define $\tilde{f}(x) = f(x) - \min f(x) \geq 0$. Then from Eq. (1),

$$0 \leq \tilde{f}_{\text{avg}}[a, b] \leq \max \tilde{f}(x)$$

or equivalently,

$$\min f(x) \leq f_{\text{avg}}[a, b] \leq \max f(x). \tag{2}$$

The coordinates corresponding to the minimum and maximum of $f(x), x_{\min}$ and $x_{\max}$, form a subinterval $[x_{\min}, x_{\max}] \in [a, b]$. Since $f(x) \in C^{[a,b]}(x)$, for $x \in [x_{\min}, x_{\max}]$ $f(x)$ must take on all values in $[\min f(x), \max f(x)]$. Therefore for some (possibly multiple) $x_0 \in [x_{\min}, x_{\max}] \in [a, b]$, $f(x_0) = f_{\text{avg}}[a, b]$.

Continuity of the functions is important, because if the functions are not continuous then this is only true for a restricted set of discontinuous functions. For example, one function that does not satisfy the MVT (for an interval containing the discontinuity) is

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}, \tag{3}$$

for any finite interval $[a, b]$ with $a < 0$ and $b > 0$ since $0 < f_{\text{avg}} = \frac{b}{b-a} < 1$ and $f(x)$ is either 0 or 1. As long as $f(x)$ is continuous (and non-singular) on an interval, then the MVT holds on that interval.

2 Leibniz rule

What we want here is a formula for the rate of change of the definite integral of a surface $f(x,t)$

$$I[f(x,t);a(x),b(x)] = \int_{a(x)}^{b(x)} f(x,t)dt \tag{4}$$

as we increase x, i.e. we want dI/dx. Here, $a(x)$ and $b(x)$ are continuous and differentiable paths in the $t-x$ plane, or two definite mappings like $t(x)$. Here it is assumed they do not intersect, and $-\infty < a(x) < b(x) < \infty$, though I believe we can show that the $a < b, \forall\, x$ condition can be relaxed. Also, we can speak to the limits $|a|, |b| \longrightarrow \infty$ as long as the surface $f(x,t)$ vanishes rapidly enough in these limits, so that the integral remains convergent.

A straightforward approach to obtain the formula is to apply the limit definition of the derivative:

$$\begin{aligned} \frac{dI(x)}{dx} &= \lim_{\Delta x \to 0} \frac{I(x+\Delta x) - I(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\int_{a(x+\Delta x)}^{b(x+\Delta x)} f(x+\Delta x, t)dt - \int_{a(x)}^{b(x)} f(x,t)dt \right) \end{aligned} \tag{5}$$

Now, we are concerned with the limiting behavior of this expression as $\Delta x \longrightarrow 0$, as it becomes a small parameter (as small as we like, but non-zero, yet). So we can perform a Taylor series expansion of both the integrand, and the limits in the first integral in the expression (5). Let us focus on that integral for now:

$$\int_{a(x+\Delta x)}^{b(x+\Delta x)} f(x+\Delta x, t)dt = \int_{a(x)+a'(x)\Delta x....}^{b(x)+b'(x)\Delta x+....} \left(f(x,t) + f'(x,t)\Delta x + \frac{1}{2!} f''(x,t)\Delta x^2 + ... \right) dt \tag{6}$$

where the primes denote partial differentiation with respect to x, $b'(x) = \partial b(x)/\partial x$.

Note that in single-variable calculus (where we suppress the $F(t)dt$),

$$\int_a^{b+r} = \int_a^b + \int_b^{b+r} \tag{7}$$

$$\int_{a+r}^{b} = \int_a^b - \int_a^{a+r}. \tag{8}$$

We can apply these rules to our expanded integral to write Eq. (6) as

$$\begin{aligned} &\int_{a(x)+a'(x)\Delta x....}^{b(x)+b'(x)\Delta x+....} \left(f(x,t) + f'(x,t)\Delta x + \frac{1}{2!} f''(x,t)\Delta x^2 + ... \right) dt \\ &= \int_{a(x)}^{b(x)} f(x+\Delta x, t)dt + \int_{b(x)}^{b(x)+b'(x)\Delta x+...} f(x+\Delta x,t)dt - \int_{a(x)}^{a(x)+a'(x)\Delta x+...} f(x+\Delta x, t)dt \\ &= \int_{a(x)}^{b(x)} f(x,t)dt + \Delta x \int_{a(x)}^{b(x)} (f'(x,t) + \frac{1}{2} f''(x,t)\Delta x + ...)dt \\ &\int_{b(x)}^{b(x)+b'(x)\Delta x+...} f(x,t)dt + \Delta x \int_{b(x)}^{b(x)+b'(x)\Delta x+...} (f'(x,t) + \frac{1}{2} f''(x,t)\Delta x + ...)dt \\ &- \int_{a(x)}^{a(x)+a'(x)\Delta x+...} f(x,t)dt - \Delta x \int_{a(x)}^{a(x)+a'(x)\Delta x+...} (f'(x,t) + \frac{1}{2} f''(x,t)\Delta x + ...)dt \end{aligned} \tag{9}$$

The first term is canceled by the subtraction in Eq. (5), and when we divide the rest by Δx and take the limit as $\Delta x \longrightarrow 0$, we have

$$\begin{aligned}\frac{dI}{dx} = \lim_{\Delta x\to 0} \frac{1}{\Delta x}\int_{b(x)}^{b(x)+b'(x)\Delta x+\ldots} f(x,t)dt - \frac{1}{\Delta x}\int_{a(x)}^{a(x)+a'(x)\Delta x+\ldots} f(x,t)dt \\ + \int_{a(x)}^{b(x)} f'(x,t)dt \\ + \int_{b(x)}^{b(x)+b'(x)\Delta x+\ldots} f'(x,t)dt - \int_{a(x)}^{a(x)+a'(x)\Delta x+\ldots} f'(x,t)dt \end{aligned} \tag{10}$$

Now, the last two terms here both vanish as $\Delta x \to 0$ as the limits of integration become nothing. Another way of seeing this is to note that these integrals are $\sim \Delta x$ and thus vanish in the limit. Furthermore, using Eq. (1), we have

$$\begin{aligned}\lim_{\Delta x\to 0} \frac{1}{\Delta x}\int_{b(x)}^{b(x)+b'(x)\Delta x+\ldots} f(x,t)dt &= \lim_{\Delta x\to 0} f_{\text{avg}}(x,[b(x),b(x)+b'(x)\Delta x+\ldots])\left(b'(x)+\frac{1}{2}b''(x)\Delta x+\ldots\right) \\ &= f(x,b(x))b'(x), \end{aligned} \tag{11}$$

with a similar result for the $a(x)$ integral. Thus we are left with

$$\begin{aligned}\frac{dI}{dx} &= \\ &= \frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = \left[f(x,b(x))\frac{db(x)}{dx} - f(x,a(x))\frac{da(x)}{dx}\right] + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x}dt \end{aligned} \tag{12}$$

Equation (12) is known as the Leibniz rule for differentiation under the integral sign.

3 Exercises regarding application of the Leibniz rule

- Use Eq. (12) to derive the Euler-Lagrange equation, by finding an extremum of I with $f(x,t) = L(y(x,t),\dot{y}(x,t);t)$ (dot being derivative w/respect to t).
- Generalize the Euler-Lagrange result to one with prescribed variation of the boundary points $a(x), b(x)$.